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HELIUM RESEARCH CENTER

INTERNAL REPORT

CORRECTION FOR NON-UNIFORMITY OF THE BORE

OF A CAPILLARY TUBE VISCOSIMETER

BY

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CORRECTION FOR NON-UNIFORMITY OF THE BORE OF A CAPILLARY TUBE VISCOSIMETER

by

J. E. Miller,[1/] R. A. Guereca,[2/]
H. P. Richardson,[1/] and J. L. Gordon[1/]

ABSTRACT

Analytic solutions for correction of non-uniformity in viscosimeter capillary bores are presented for the following cases: ellipse, cone, sine wave, sawtooth wave, square wave, and the general case. Rapid, accurate estimates of the correction factor $(1+\alpha)$ may be obtained from a graph of percent relative deviation in the bore versus α. Comparison of the analytic solutions to previously published data on glass capillary tubes illustrates the convenience of the analytic solutions. The general case is used to estimate $(1+\alpha)$ for a section of stainless steel capillary tubing. The radius of a glass capillary may be determined by filling the bore with mercury and measuring the electrical resistance. The correction factor for non-uniformity of the bore is shown to be $\alpha/3$ using the resistance method.

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INTRODUCTION

At a given temperature (T) and pressure (P), for a steady-state laminar volumetric flowrate, Q, through a capillary of length, L_T, and resulting pressure drop, ΔP, the Poiseuille equation (1) for viscosity, η, is derived,

$$\eta = \left[\frac{\pi \Delta P r^4}{8 Q L_T} \right]_{(T,P)} , \qquad (1)$$

by assuming that the tube bore is a perfect right circular cylinder with radius r. Because real capillary bores are non-uniform, a correction is applied to equation (1). This correction is computed by assuming that if deviations in the bore are small, then the pressure drop, dP_i, over length, dx_i, will still be proportional to $1/r_i^4$ because the radial component of velocity may be considered negligible (1).[3/]

3/ Underlined numbers in parentheses refer to items in the list of references at the end of this report.

If this is true, $(L_T/r^4)^{-1}$ in (1) can be replaced by

$$\left[\int_{x_i=0}^{x_i=L_T} \frac{dx_i}{r_i^4} \right]^{-1}$$

or r^4 in (1) can be replaced by

$$\left[\frac{1}{L_T} \int_{x_i=0}^{x_i=L_T} \frac{dx_i}{r_i^4} \right]^{-1} . \tag{2}$$

When the radial component of velocity is not negligible, the actual velocity distribution must be determined and equation (1) rederived because dP_i/dx_i along the cylinder axis will be perturbed. In this report, all calculations of the correction for non-uniformity of the bore are based on the assumption that the expression in (2) is valid.

In most of the previous work with absolute capillary flow viscosimeters, a root-mean-square radius, r_{ms}, is determined by measuring the internal volume and length of the capillary; then r_{ms} of a right circular cylinder with the same length and volume is computed. The dimensionless correction for non-uniformity of the bore (δ) is defined so that

$$\frac{r_{ms}^4}{\delta} = \left[\frac{1}{L_T} \int_{x_i=0}^{x_i=L_T} \frac{dx_i}{r_i^4} \right]^{-1} \tag{3}$$

or

$$\delta = \left[\frac{r_{ms}^4}{L_T} \int_{x_i=0}^{x_i=L_T} \frac{dx_i}{r_i^4} \right] . \tag{4}$$

As a consequence of equation (3), the viscosity equation (1) assumes the form

$$\eta = \frac{\pi \Delta P r_{ms}^4}{8QL_T\delta} . \tag{1a}$$

For a right circular cylinder, $\delta = 1$; when $r_i \neq$ a constant, $\delta > 1$. As is well known, the pressure drop through an actual capillary bore is greater than would be observed in the corresponding perfect cylinder.

There are a few cases where equation (4) may be solved exactly. Barr (1) gives the solutions for an ellipse, cone, and cone with elliptical cross sections. In this report, solutions for a sine wave, sawtooth wave, square wave, and the general case are presented, together with those for the cone and ellipse. All solutions are reduced to the same form:

$$\delta - 1 = \alpha = f(K) , \tag{5}$$

where K is roughly the relative deviation from symmetry. Figure 1 is a

Figure 1.-Graphs of the bore profiles.

graphic representation of the bore profiles.

EXACT SOLUTIONS OF EQUATION (4)

Ellipse

In this case, the semi-axes r_{max} and r_{min} are at right angles and there is no rotation of axes down the length of the bore. The area of a cross section is $\left[\pi\, r_{max}\, r_{min}\right]$, so r_m is the radius of a circle having the same area:

$$\pi\, r_m^2 = \pi\, r_{max}\, r_{min} \; ; \tag{6}$$

$$r_m^4 = r_{max}^2\, r_{min}^2 \; . \tag{7}$$

According to Barr (1), evaluation of (2) gives

$$\frac{2\, r_{max}^3\, r_{min}^3}{r_{max}^2 + r_{min}^2} \; , \tag{8}$$

so equation (4) becomes:

$$\delta \equiv 1 + \alpha_E = r_{max}^2\, r_{min}^2 \left(\frac{r_{max}^2 + r_{min}^2}{2\, r_{max}^3\, r_{min}^3} \right) . \tag{9}$$

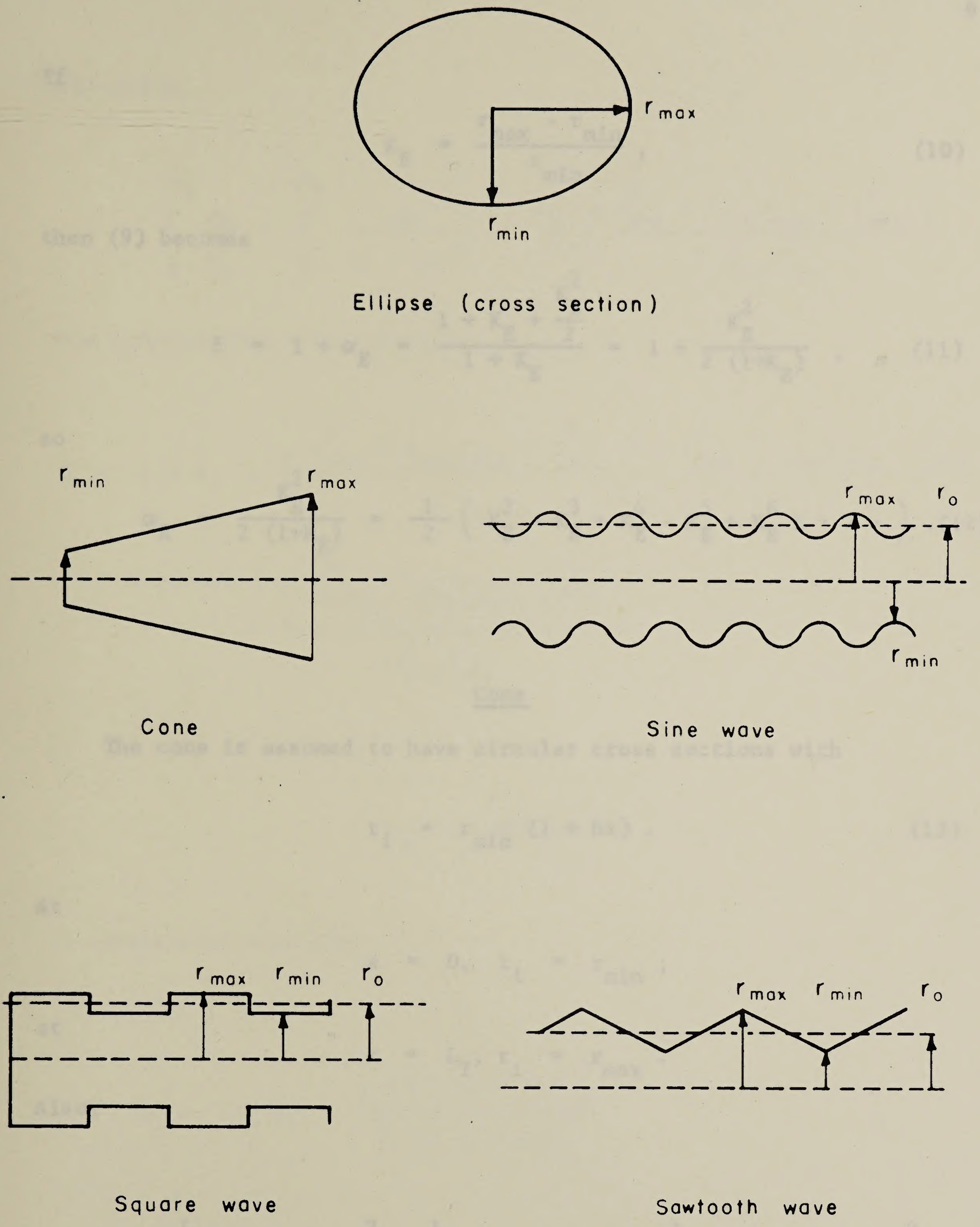

FIGURE 1.- Graphs of the Bore Profiles.

If

$$K_E = \frac{r_{max} - r_{min}}{r_{min}} , \qquad (10)$$

then (9) becomes

$$\delta = 1 + \alpha_E = \frac{1 + K_E + \frac{K_E^2}{2}}{1 + K_E} = 1 + \frac{K_E^2}{2\,(1+K_E)} , \qquad (11)$$

so

$$\alpha_E = \frac{K_E^2}{2\,(1+K_E)} = \frac{1}{2}\left(K_E^2 - K_E^3 + K_E^4 - K_E^5 + K_E^6 \pm \cdot \cdot \cdot \right) . (12)$$

Cone

The cone is assumed to have circular cross sections with

$$r_i = r_{min}\,(1 + bx) . \qquad (13)$$

At

$$x = 0,\ r_i = r_{min} ;$$

at

$$x = L_T,\ r_i = r_{max} .$$

Also,

$$r_m^2 = \frac{1}{L_T}\int_0^{L_T} r_i^2\,dx = \frac{r_{min}^2}{L_T}\int_0^{L_T} (1 + bx)^2 = \frac{r_{max}^2 + r_{max}r_{min} + r_{min}^2}{3} . \qquad (14)$$

Evaluating,

$$\int_0^{L_T} \frac{dx}{r_i^4} = \int_0^{L_T} \frac{dx}{(1 + bx)^4} = \frac{L_T}{3} \left[\frac{r_{max}^2 + r_{max} r_{min} + r_{min}^2}{3\, r_{max}^3 r_{min}^3} \right]. \quad (15)$$

Then substitution into (4) gives

$$\delta = 1 + \alpha_C = \left[\frac{r_{max}^2 + r_{max} r_{min} + r_{min}^2}{3} \right]^2 \left[\frac{r_{max}^2 + r_{max} r_{min} + r_{min}^2}{3\, r_{max}^3 r_{min}^3} \right]$$

$$= \frac{\left[\frac{r_{max}^2 + r_{max} r_{min} + r_{min}^2}{3} \right]^3}{r_{max}^3 r_{min}^3} . \quad (16)$$

If

$$K_C = \frac{r_{max} - r_{min}}{r_{min}} \quad (17)$$

is substituted into (16):

$$\delta = 1 + \alpha_C = \frac{\left[(1 + K_C)^3 - 1 \right]^3}{27\, K_C^3 (1 + K_C)^3}$$

$$= 1 + \frac{K_C^2}{1 + K_C} + \frac{K_C^4}{3\,(1 + K_C)^2} + \frac{K_C^6}{27\,(1 + K_C)^3} \quad (18)$$

and

$$\alpha_C = \frac{K_C^2}{1 + K_C} + \frac{K_C^4}{3\,(1 + K_C)^2} + \frac{K_C^6}{27\,(1 + K_C)^3}$$

$$= K_C^2 - K_C^3 + \frac{4}{3} K_C^4 - \frac{15}{9} K_C^5 + \frac{55}{27} K_C^6 - \frac{22}{9} K_C^7 \pm \cdot \cdot \cdot \qquad (20)$$

Sine Wave[4/]

4/ K. R. Van Doren, research mathematician, Helium Research Center, Bureau of Mines, Amarillo, Tex., contributed substantially in developing this solution.

The equation of a sine wave deviation superimposed on a right circular cylinder is

$$r_i = r_0 + C_1 \sin C_2 x \,. \qquad (21)$$

Then

$$r_m^2 = \frac{1}{L_T} \int_0^{L_T} (r_0 + C_1 \sin C_2 x)^2 \, dx \qquad (22)$$

$$= r_0^2 + \frac{2\, r_0 C_1}{C_2\, L_T} (1 - \cos C_2\, L_T) + \frac{C_1^2}{2} - \frac{C_1^2}{4\, C_2\, L_T} \sin 2\, C_2\, L_T \,, \qquad (23)$$

and for an integral number of cycles,

$$r_m^2 = r_0^2 + \frac{C_1^2}{2} . \tag{24}$$

The other integral,

$$\int_0^{L_T} \frac{dx}{(r_0 + C_1 \sin C_2 x)^4} , \tag{25}$$

can be solved by making the following substitutions:

$$u = C_2 x ; \tag{26}$$

$$Z = \tan \frac{u}{2} . \tag{27}$$

Then

$$\int_0^{L_T} \frac{dx}{(r_0 + C_1 \sin C_2 x)^4} = \frac{2}{C_2} \int_0^{Z = \tan \frac{C_2 L_T}{2}} \frac{(1 + Z^2)^3 \, dZ}{(r_0 + 2 C_1 Z + r_0 Z^2)^4} \tag{28}$$

and the numerator can be expanded to give four integrals that can be solved by standard methods, although each of the four must be broken up into two parts to prevent integration across a discontinuity. The answer to (28) is a very long expression, but for an integral number of cycles it is, after making the substitution:

$$K_S = \frac{C_1}{r_0} = \frac{r_{max} - r_0}{r_0} = \frac{r_0 - r_{min}}{r_0} , \tag{29}$$

$$\int_0^{L_T} \frac{dx}{(r_0 + C_1 \sin C_2 x)^4} = \frac{L_T (2 + 3K_S^2)}{2 r_0^4 (1 - K_S^2)^{7/2}} . \tag{30}$$

Substituting (29), (30), and (24) into (4) gives:

$$\delta = 1 + \alpha_S = \frac{(1 + \frac{K_S^2}{2})^2 (2 + 3K_S^2)}{(1 - K_S^2)^{7/2}} . \tag{31}$$

Equation (31) can be simplified to give

$$\delta = 1 + \alpha_S = 1 + 6 K_S^2 + 18.375 K_S^4 + 29.375 K_S^6 + \cdot \cdot \cdot \tag{32}$$

<u>Sawtooth Wave</u>

A sawtooth wave superimposed on a cylinder may be solved using the same techniques. If K_T is defined as

$$K_T = \frac{r_{max} - r_0}{r_0} = \frac{r_0 - r_{min}}{r_0} , \tag{33}$$

it can be shown that

$$r_m^2 = r_0^2 (1 + \frac{K_T^2}{3}) \tag{34}$$

and

$$\delta = 1 + \alpha_T = \frac{r_m^4}{L_T} \int_0^{L_T} \frac{dx}{r_i^4} = \frac{(1 + \frac{K_T^2}{3})^3}{(1 - K_T^2)^3} \tag{35}$$

or

$$\delta = 1 + \alpha_T = 1 + 4 K_T^2 + \frac{28}{3} K_T^4 + \frac{460}{27} K_T^6 + \frac{244}{9} K_T^8 + \cdot \cdot \cdot \tag{36}$$

Square Wave

If the square wave deviation is symmetrical to r_0,

$$r_i = r_0 (1 \pm K_W) \tag{37}$$

and

$$r_m^2 = \frac{1}{L_T} \left[\int_0^{\frac{L_T}{2}} r_0^2 (1 + K_W)^2 dx + \int_{\frac{L_T}{2}}^{L_T} r_0^2 (1 - K_W)^2 \right]$$

$$= r_0^2 (1 + K_W^2) \tag{38}$$

for an even number of cycles. Also,

$$\frac{1}{L_T} \int_0^{L_T} \frac{dx}{r_i^4} = \frac{1}{L_T} \left[\frac{L_T}{2 r_0^4 (1 + K_W)^4} + \frac{L_T}{2 r_0^4 (1 - K_W)^4} \right] . \tag{39}$$

Then

$$\delta = 1 + \alpha_W = \frac{(1 + K_W^2)^2}{2}\left[\frac{(1 - K_W)^4 + (1 + K_W)^4}{(1 - K_W)^4(1 + K_W)^4}\right]$$

$$= \frac{(1 + K_W^2)^2(1 + 6\,K_W^2 + K_W^4)}{(1 - K_W^2)^4}$$

$$= 1 + 12\,K_W^2 + 56\,K_W^4 + 164\,K_W^6 + 368\,K_W^8 + \cdots \qquad (40)$$

Table 1 summarizes analytical solutions for all cases discussed. Table 2 shows values for α for various values of K. Figure 2 shows a

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Figure 2.-Values for K versus α.

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graphic representation of K versus α.

General Case

In some cases it may not be possible to find an analytic expression of $r_i = f(x)$ for substitution into (4). Then (2) must be evaluated by graphical or numerical methods. Swindells, Coe, and Godfrey (5) replaced the integral by a summation. Using our notation, they let

$$\frac{1}{L_T}\int_0^{L_T} \frac{dx}{r_i^4} = \frac{1}{n}\sum_1^n \frac{1}{r_i^4} \qquad (41)$$

TABLE 1.-Summary of analytic solutions for $\delta - 1 = \alpha = f(K)$

Type of bore[1/]	K	$\delta - 1 = \alpha$
Ellipse	$\frac{r_{max} - r_{min}}{r_{min}}$	$\frac{1}{2}\left(K^2 - K^3 + K^4 - K^5 + K^6 \pm \cdot \cdot \cdot\right)$
Cone	$\frac{r_{max} - r_{min}}{r_{min}}$	$K^2 - K^3 + \frac{4}{3} K^4 - \frac{15}{9} K^5 + \frac{55}{27} K^6 \pm \cdot \cdot \cdot$
Sine wave	$\frac{r_{max} - r_0}{r_0}$	$6 K^2 + 18.375 K^4 + 29.375 K^6 + \cdot \cdot \cdot$
Sawtooth wave	$\frac{r_{max} - r_0}{r_0}$	$4 K^2 + \frac{28}{3} K^4 + \frac{460}{27} K^6 + \cdot \cdot \cdot$
Square wave	$\frac{r_{max} - r_0}{r_0}$	$12 K^2 + 56 K^4 + 164 K^6 + \cdot \cdot \cdot$
General	$\frac{r_i - r_m}{r_m}$	$- 24 K_{avg} \pm \cdot \cdot \cdot$, or $24 \left(\frac{r_m - r_{avg}}{r_m}\right)$

1/ See figure 1.

TABLE 2.-Values for δ-1 = α = f(K)

K, %[1/]	δ-1 = α = f(K), 10^{-6} units				
	Ellipse, α_E	Cone, α_C	Sawtooth wave, α_T	Sine wave, α_S	Square wave, α_W
0.01	[2/]0.005	0.01	0.04	0.06	0.12
.02	.02	.04	.16	.24	.48
.04	.08	.16	.64	.96	1.92
.06	.18	.36	1.44	2.16	4.32
.06	.32	.64	2.56	3.84	7.68
.10	.50	1.00	4.00	6.00	12.00
.12	.72	1.44	5.76	8.64	17.28
.14	.98	1.96	7.84	11.76	23.52
.16	1.28	2.56	10.24	15.36	30.72
.18	1.62	3.23	12.96	19.44	38.94
.20	2.00	3.99	16.00	24.00	48.00
.22	2.42	4.83	19.36	29.04	58.21
.24	2.88	5.75	23.04	34.56	69.31
.26	3.37	6.74	27.04	40.56	81.38
.28	3.91	7.82	31.36	47.04	94.42
.30	4.49	8.97	36.00	54.00	108.45
.4	8	16	64	96	192
.6	18	36	144	216	432
.8	32	63	256	384	768
1.0	50	99	400	600	1,200
1.2	71	142	576	864	1,728
1.4	96	193	784	1,177	2,352
1.6	126	252	1,024	1,538	3,072
1.8	159	318	1,297	1,946	3,894
2.0	196	392	1,601	2,403	4,800
2.2	237	474	1,938	2,908	5,821
2.4	281	562	2,307	3,462	6,931
2.6	330	659	2,708	4,064	8,138
2.8	382	763	3,142	4,715	9,442
3.0	437	874	3,608	5,415	10,845

1/ See table 1 for definition of K.
2/ $\alpha = 0.005 \times 10^{-6}$, $\delta = 1.000000005$.

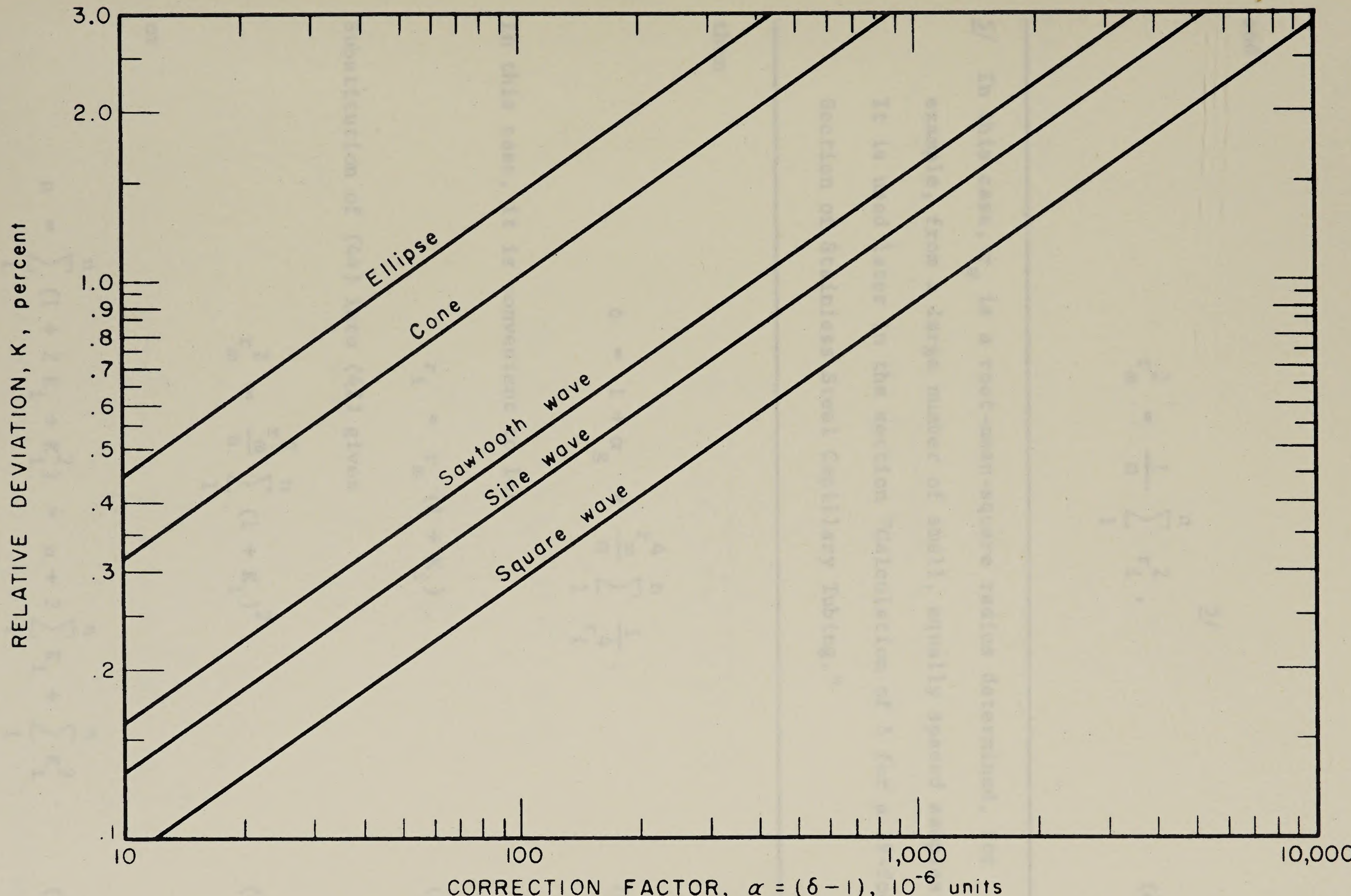

FIGURE 2.-Values for K Verus α.

and

$$r_m^2 = \frac{1}{n} \sum_{1}^{n} r_i^2 , \quad \text{5/} \tag{42}$$

5/ In this case, r_m is a root-mean-square radius determined, for example, from a large number of small, equally spaced samples, n. It is used later in the section "Calculation of δ for a 19-foot Section of Stainless Steel Capillary Tubing."

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then

$$\delta = 1 + \alpha_g = \frac{r_m^4}{n} \sum_{1}^{n} \frac{1}{r_i^4} . \tag{43}$$

In this case, it is convenient to let

$$r_i = r_m (1 + K_i) . \tag{44}$$

Substitution of (44) into (42) gives

$$r_m^2 = \frac{r_m^2}{n} \sum_{1}^{n} (1 + K_i)^2 \tag{45}$$

or

$$n = \sum_{1}^{n} (1 + 2 K_i + K_i^2) = n + 2 \sum_{1}^{n} K_i + \sum_{1}^{n} K_i^2 \tag{46}$$

and

$$2 \sum_{1}^{n} K_i + \sum_{1}^{n} K_i^2 = 0 . \tag{47}$$

Substitution of (44) into (43) gives

$$\delta = 1 + \alpha_g = \frac{r_m^4}{n} \sum_{1}^{n} \frac{1}{r_m^4 (1 + K_i)^4} = \frac{1}{n} \sum_{1}^{n} \frac{1}{(1 + K_i)^4}$$

$$= \frac{1}{n} \sum_{1}^{n} \frac{1}{1 + 4 K_i + 6 K_i^2 + 4 K_i^3 + K_i^4}$$

$$= \frac{1}{n} \sum_{1}^{n} (1 - 4 K_i + 10 K_i^2 - 20 K_i^3$$

$$+ 35 K_i^4 - 56 K_i^5 + 84 K_i^6$$

$$- 120 K_i^7 \pm \cdot \cdot \cdot)$$

$$= \frac{1}{n} \Big[n - 4 \sum_{1}^{n} K_i + 10 \sum_{1}^{n} K_i^2 - 20 \sum_{1}^{n} K_i^3$$

$$+ 35 \sum_{1}^{n} K_i^4 \pm \cdot \cdot \cdot \Big] . \tag{48}$$

From (47),

$$- 4 \sum_{1}^{n} K_i = 2 \sum_{1}^{n} K_i^2$$

so

$$\delta = 1 + \alpha_g = \frac{1}{n} \left[n + 12 \sum_{1}^{n} K_i^2 - 20 \sum_{1}^{n} K_i^3 + 35 \sum_{1}^{n} K_i^4 \pm \cdot \cdot \cdot \right] . \quad (49)$$

The average K_i is

$$K_{avg} = \frac{1}{n} \sum_{1}^{n} K_i = - \frac{1}{2n} \sum_{1}^{n} K_i^2 \quad (50)$$

so

$$\sum_{1}^{n} K_i^2 = - 2n K_{avg} . \quad (51)$$

Substituting (51) into (49) gives

$$\delta = 1 + \alpha_g = 1 - 24 K_{avg} - \frac{20}{n} \sum_{1}^{n} K_i^3 + \frac{35}{n} \sum_{1}^{n} K_i^4 \pm \cdot \cdot \cdot$$

$$\delta \cong 1 - 24 K_{avg} , \quad (52)$$

which is a very good approximation because K_i^3 is usually negligible and is both plus and minus so the sum tends to cancel. Then the sum of

K_i^3 is negligible also. Equation (52) is essentially the same as equation (IV-8) in Flynn (2).

A more convenient equation to compute δ can be derived from (50):

$$K_{avg} = \frac{\sum_{1}^{n} r_i}{n\, r_m} - 1 . \tag{53}$$

But

$$r_{avg} = \frac{1}{n} \sum_{1}^{n} r_i , \tag{54}$$

so

$$K_{avg} = \frac{n\, r_{avg}}{n\, r_m} - 1 = \frac{r_{avg} - r_m}{r_m}$$

and

$$\delta \cong 1 - 24\, K_{avg} = 1 + \frac{24\,(r_m - r_{avg})}{r_m} = 1 + \frac{12\sigma^2}{r_m^2} , \tag{55}$$

where

$$\sigma = \left[\frac{1}{n} \sum_{1}^{n} (r_i - r_{avg})^2 \right]^{\frac{1}{2}} .$$

Equation (41) is essentially a trapezoidal rule integration

$$\int_a^b y\, dx \cong h \left(\frac{1}{2} y_0 + y_1 + y_2 + \cdot \cdot \cdot \; y_{n-1} + \frac{1}{2} y_n \right) ;$$

$$h = \frac{b - a}{n} .$$

So

$$\frac{1}{L_T} \int_0^{L_T} \frac{dx}{r_i^4} = \frac{L_T}{n\,L_T} \left(\frac{1}{2r_0^4} + \frac{1}{r_1^4} + \frac{1}{r_2^4} + \cdot \cdot \cdot \frac{1}{r_{n-1}^4} + \frac{1}{2r_n^4} \right) ;$$

and if $r_0 \cong r_n$, or choose $r_0 = r_n$,

$$\frac{1}{L_T} \int_0^{L_T} \frac{dx}{r_i^4} = \frac{1}{n} \left(\frac{1}{r_1^4} + \frac{1}{r_2^4} + \cdot \cdot \cdot \frac{1}{r_n^4} \right) = \frac{1}{n} \sum_1^n \frac{1}{r_i^4} .$$

Also, (44) is derived from

$$r_m^2 = \frac{1}{L_T} \int_0^{L_T} r_i^2\, dx \cong \frac{L_T}{nL_T} \left(\frac{r_0^2}{2} + r_1^2 + r_2^2 + \cdot \cdot \cdot r_{n-1}^2 + \frac{r_n^2}{2} \right)$$

$$r_m^2 \cong \frac{1}{n} \sum_1^n r_i^2 .$$

These equations are for n equally spaced samples. One must know the exact profile of $r_i = f(x)$ to get an "exact" value for either r_m or δ. Simpson's rule gives nearly the same result as the trapezoidal rule in the evaluation of (42) and (43).

Due to the way K_i is defined, K_{avg} will always be negative, so δ is always greater than one. It is necessary to use r_m computed from

(42) to get a valid estimate of δ from (43). Otherwise, the estimate for δ will be in error by the ratio of the two r's raised to the fourth power and it is possible to compute values of δ that are less than one. The use of (42) and (43) with relative r_i measurements will give a valid estimate for δ; this is essentially the procedure used by Swindells, Coe, and Godfrey (5).

It is interesting to note that one cannot let K_i be constant in this derivation. Because of the way K_i is defined, (47) would give:

$$2\,n\,K + n\,K^2 = 0 \qquad (56)$$

or

$$K\,(K + 2) = 0$$

and

$$K = 0,\ -2\ .$$

The first case corresponds to $r_i = r_m$, which is a perfect cylinder. The second case gives $r_i = -r_m$, which is impossible. If $K_i = \pm$ constant, one should use (40) for the square wave.

SAMPLE CALCULATIONS

Four examples are given to illustrate the use of equations and figure 2 in this report. Some relative diameter measurements given in (5) are used for the first three examples. These measurements are shown

in table 3 and are graphed in figure 3; the deviations are roughly

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Figure 3.-Cross-sectional area of bore at regular positions along the capillaries.

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sinusoidal with

$$K_S = \frac{r_{max} - r_0}{r_0} \cong \frac{562.6 - 561.625}{561.625} \cong 0.00174 , \tag{57}$$

$$= 0.174 \text{ percent} .$$

From the sine wave curve of figure 2, $\alpha = 18 \times 10^{-6}$ and $\delta = 1.000018$; (5) gives $\delta = 1.000021$.

The data in table 3 may be substituted into equations (42) and (43) giving $\delta = 1.00002086$; using the approximation of (52) and (53) yields

$$K_{avg} = \frac{6188.625}{6188.630382} - 1 = -0.86966 \times 10^{-6} \tag{58}$$

and

$$\delta = 1 - 24 K_{avg} = 1.00002087, \tag{59}$$

so the approximate equation gives very good results in this case.

The third example is for capillary tube number 2.5a of (5); cross-sectional areas are seen to be similar to a sawtooth deviation. The radius is proportional to the square root of area, so K_T may be taken as:

TABLE 3.-Variations in cross-sectional area along capillary 2.5, arbitrary units

Position along tube	Mean diameter	Area of cross section
cm		
1	1,123.75	126 281
5	1,123.25	126 169
10	1,124.75	126 506
15	1,127.25	127 069
20	1,125.75	126 731
25	1,125.00	126 563
30	1,125.00	126 563
35	1,124.25	126 394
40	1,127.75	127 182
45	1,127.00	127 013
49	1,123.50	126 225

Source: Swindells, J. G., J. R. Coe, and T. B. Godfrey. Absolute Viscosity of Water at 20° C. NBS J. Res., v. 48, No. 1, January 1952, p. 16.

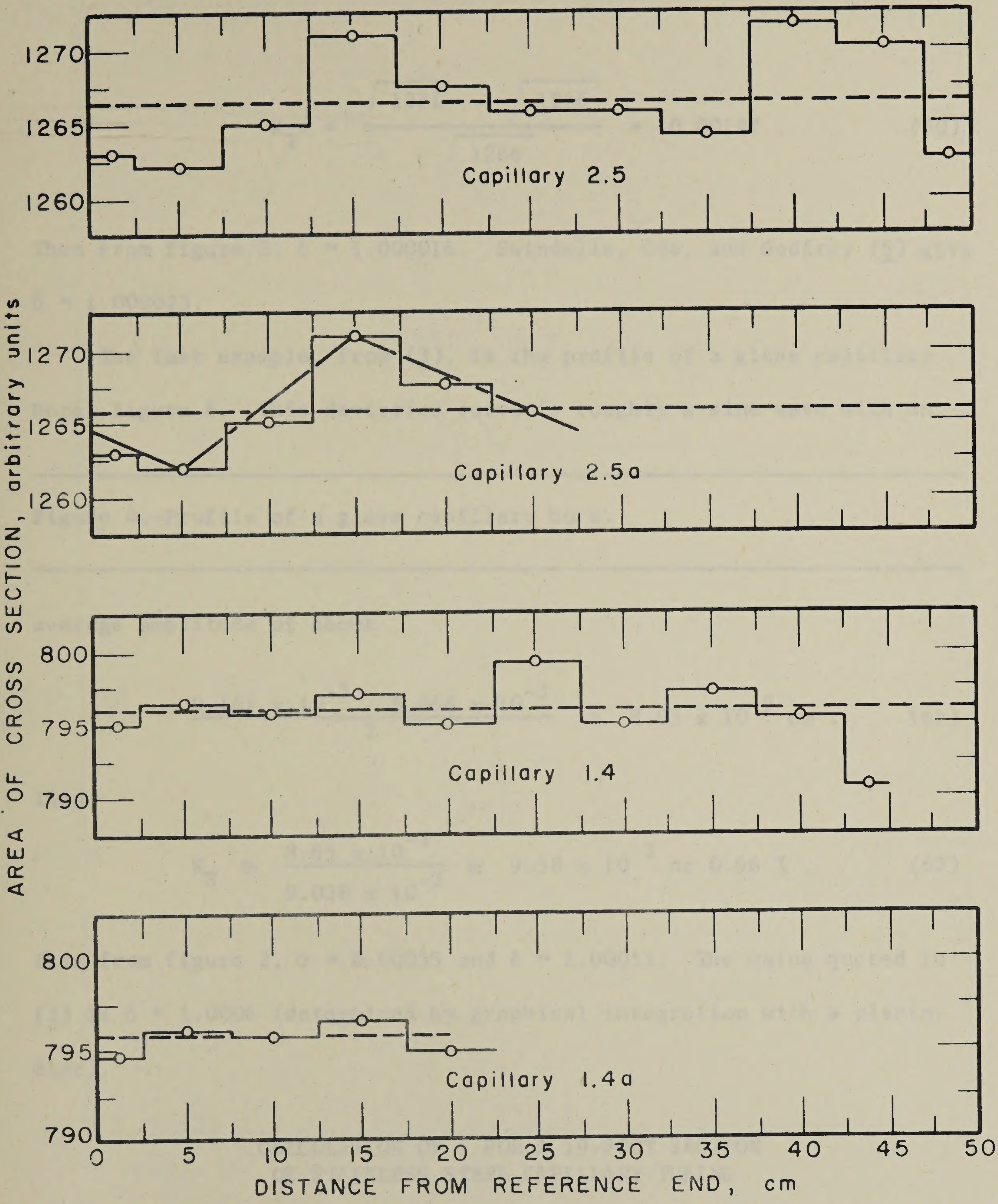

FIGURE 3.– Cross-Sectional Area of Bore at Regular Positions Along the Capillaries.

Source: Swindells, J. G., J. R. Coe, and T. B. Godfrey. Absolute Viscosity of Water at 20° C. NBS J. Res., v. 48, No. 1, January 1952, p. 16.

$$K_T = \frac{\sqrt{1271} - \sqrt{1266}}{\sqrt{1266}} = 0.00197 \; . \qquad (60)$$

Then from figure 2, δ = 1.000016. Swindells, Coe, and Godfrey (5) give δ = 1.000023.

The last example, from (3), is the profile of a glass capillary bore, figure 4. This deviation curve is roughly a sine wave with an

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Figure 4.-Profile of a glass capillary bore.

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average amplitude of about

$$\frac{9.141 \times 10^{-3} - 8.968 \times 10^{-3}}{2} = 8.65 \times 10^{-5} \text{ cm} \; . \qquad (61)$$

Then

$$K_S \cong \frac{8.65 \times 10^{-5}}{9.028 \times 10^{-3}} \cong 9.58 \times 10^{-3} \text{ or } 0.96\ \% \; . \qquad (62)$$

Then from figure 2, α = 0.00055 and δ = 1.00055. The value quoted in (3) is δ = 1.0006 (determined by graphical integration with a planimeter).

CALCULATION OF δ FOR A 19-FOOT SECTION OF STAINLESS STEEL CAPILLARY TUBING

A 19-foot section of 347 stainless steel capillary tubing supplied by Superior Tube Company,[6/] Norristown, Pa., was cut from one

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6/ Trade names are used for identification only and endorsement by the

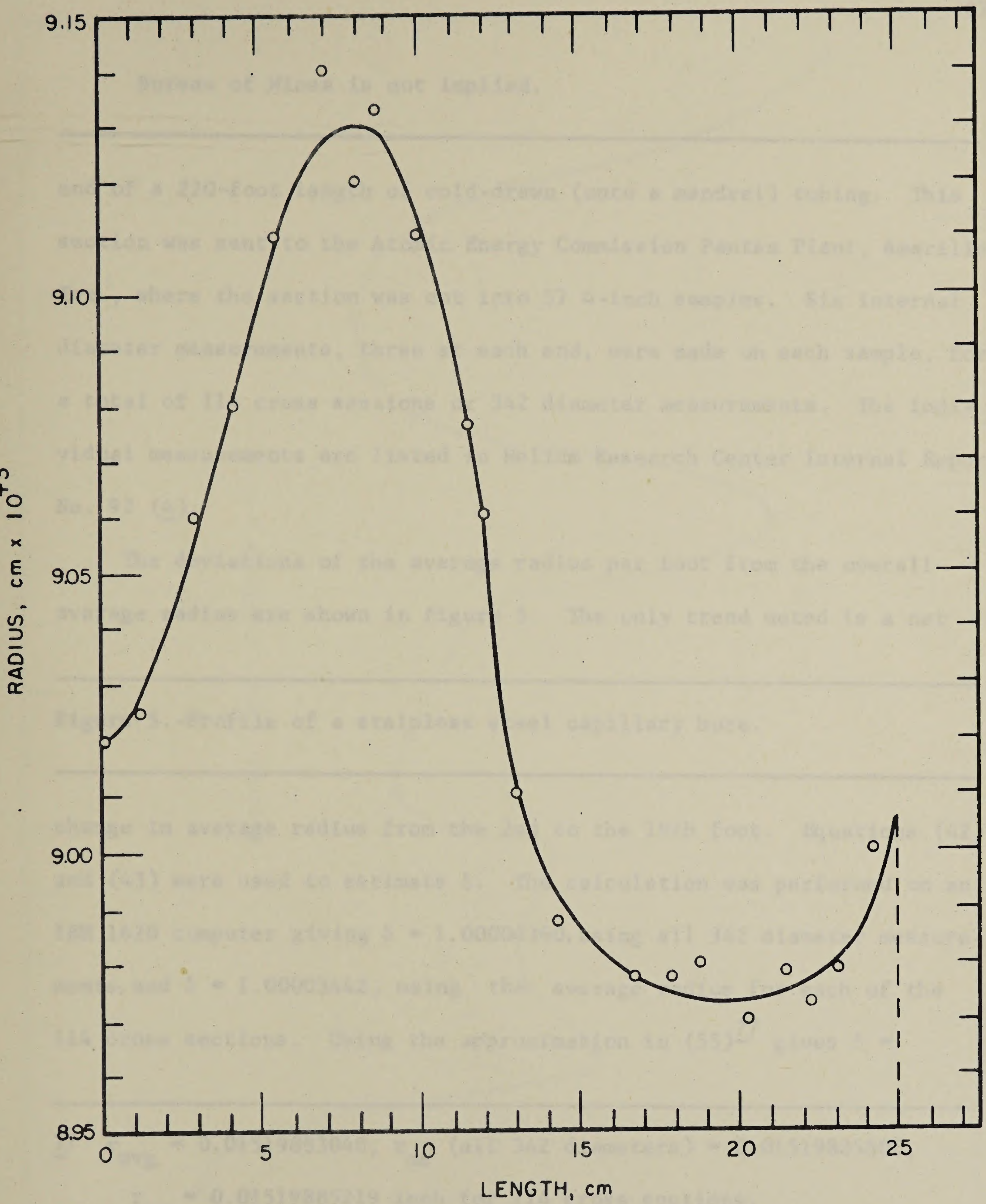

FIGURE 4.- Profile of a Glass Capillary Bore.

Source: Giddings, John G. The Viscosity of Light Hydrocarbon Mixtures at High Pressures: The Methane-Propane System. Ph. D. thesis, William Marsh Rice University, Houston, Texas, May 1963, 94+ pp. Copyright 1963.

Bureau of Mines is not implied.

end of a 220-foot length of cold-drawn (onto a mandrel) tubing. This section was sent to the Atomic Energy Commission Pantex Plant, Amarillo, Tex., where the section was cut into 57 4-inch samples. Six internal diameter measurements, three at each end, were made on each sample, for a total of 114 cross sections or 342 diameter measurements. The individual measurements are listed in Helium Research Center Internal Report No. 92 (4).

The deviations of the average radius per foot from the overall average radius are shown in figure 5. The only trend noted is a net

Figure 5.-Profile of a stainless steel capillary bore.

change in average radius from the 2nd to the 19th foot. Equations (42) and (43) were used to estimate δ. The calculation was performed on an IBM 1620 computer giving $\delta = 1.00004360$, using all 342 diameter measurements, and $\delta = 1.00003442$, using the average radius for each of the 114 cross sections. Using the approximation in (55)[7/] gives $\delta =$

7/ $r_{avg} = 0.01519883040$, r_{ms} (all 342 diameters) $= 0.01519885800$, $r_{ms} = 0.01519885219$ inch for 114 cross sections.

1.00004358 for all 342 measurements and $\delta = 1.00003441$ for the 114 cross sections. From figure 5,

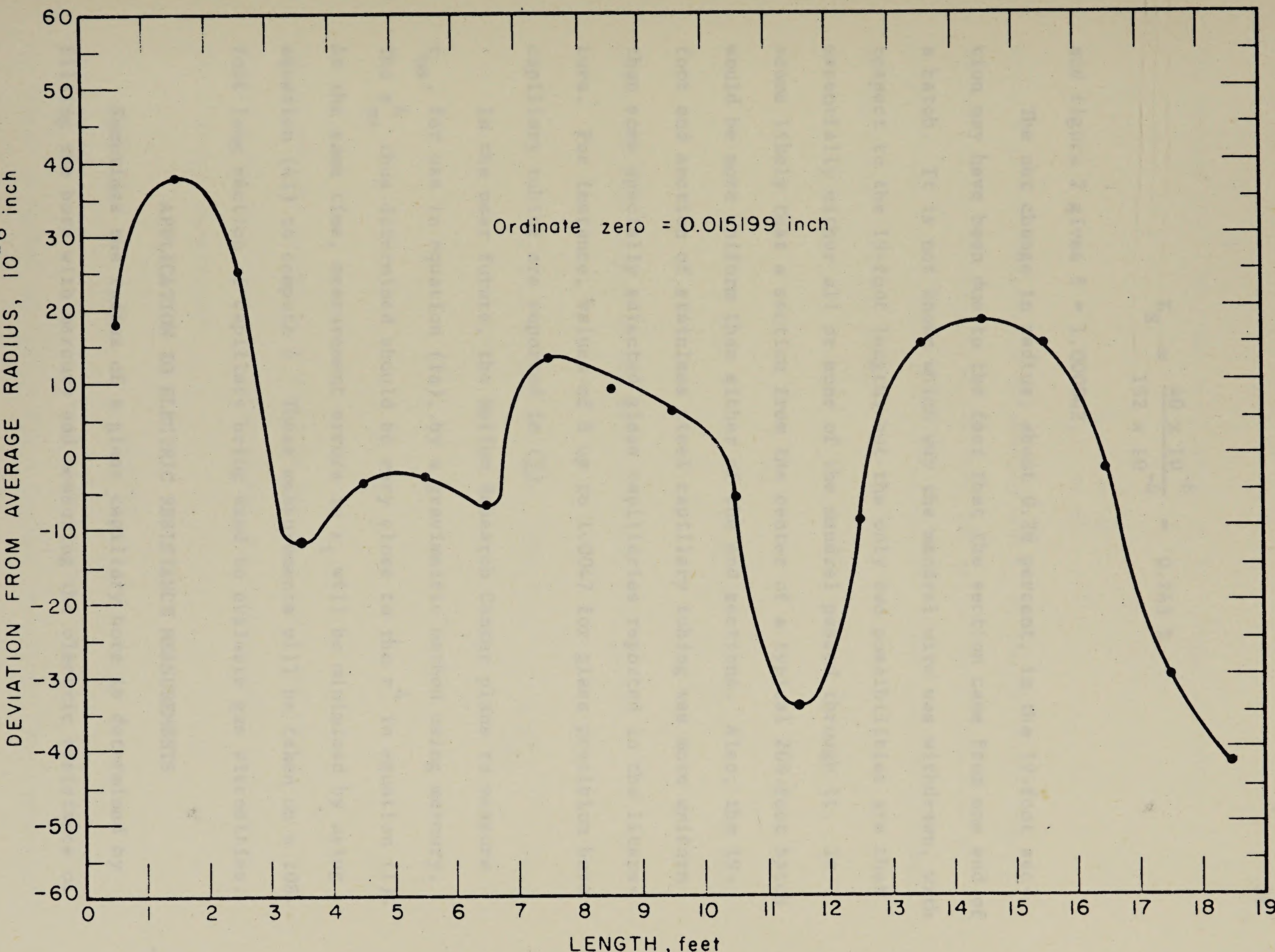

FIGURE 5.- Profile of a Stainless Steel Capillary Bore.

$$K_S \cong \frac{40 \times 10^{-6}}{152 \times 10^{-4}} = 0.263 \%$$

and figure 2 gives $\delta = 1.000042$.

The net change in radius, about 0.26 percent, in the 19-foot section may have been due to the fact that the section came from one end of a batch. It is not known which way the mandrel wire was withdrawn, with respect to the 19-foot length, but the only two possibilities are that essentially either all or none of the mandrel passed through it. It seems likely that a section from the center of a typical 200-foot batch would be more uniform than either of the end sections. Also, the 19-foot end section of stainless steel capillary tubing was more uniform than some specially selected glass capillaries reported in the literature. For instance, values of δ up to 1.0047 for glass precision bore capillary tubing are reported in (3).

In the near future, the Helium Research Center plans to measure r_{ms}, for use in equation (1a), by a gravimetric method using mercury. The r_{ms}^4 thus determined should be very close to the r^4 in equation (1). At the same time, measurement errors in r_i will be minimized by using equation (43) to compute δ. These measurements will be taken on a 208-foot long section of capillary being used to evaluate gas viscosities.

APPLICATION TO ELECTRIC RESISTANCE MEASUREMENTS

Sometimes the radius of a glass capillary bore is determined by filling the bore with mercury and measuring the electric resistance of

the mercury (5). Then the resistance, R, is related to radius, r, by

$$R = \frac{\rho L_T}{\pi r^2}, \tag{63}$$

where ρ is the resistivity of mercury and L_T is length of the bore. Because of non-uniformity of the bore the actual resistance will be greater than the resistance of the equivalent right circular cylinder by the factor $(1 + \triangle)$, where

$$1 + \triangle = \frac{r_m^2}{L_T} \int_0^{L_T} \frac{dx}{r_i^2} \cong \frac{r_m^2}{n} \sum_1^n \frac{1}{r_i^2}. \tag{64}$$

Using the same method as was used to obtain (55) gives:

$$1 + \triangle = 1 + 8 \left(\frac{r_m - r_{avg}}{r_m} \right) = 1 - 8 K_{avg}, \tag{65}$$

and

$$\triangle \cong \frac{1}{3} \alpha . \tag{66}$$

Therefore, the values of α given in tables 1 and 2 and shown in figure 2 may be divided by three to get the corresponding correction for electric resistance.

SUMMARY

In this report, all of the analytic solutions for δ are independent of length or number of cycles for a regular deviation. The correction

factor, δ, usually will be negligible; deviations of 0.2 percent in the root-mean-square radius lead to corrections of less than 0.005 percent in computing the viscosity, η. For relatively large deviations, the equations in this report may lead to incorrect estimates of δ because the radial component of velocity may not be negligible.

The graph of percent relative deviation, K, versus α makes it possible to obtain rapid, accurate estimates of δ. The most conservative estimate is given by the square wave function; the most realistic estimate probably is given by the sine wave deviation.

If one can replace r^4 in equation (1) by the quantity in equation (2), it is not necessary to compute δ. On the other hand, if one has measured a value for r_{ms}, an estimate for δ may be computed from equation (43) or (55) but in both cases it is necessary to use r_m computed from equation (42) to get a valid estimate for δ. The estimated value for δ can then be applied to equation (1) with r_{ms}^4/δ substituted for r^4.

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